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# On a determinant result of I. Olkin

Marvin Marcus

*2937 Kenmore Place, Santa Barbara, California 93105-2223, USA*

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**Abstract**

In a recent interesting paper [I. Olkin, Linear Algebra Appl. 264 (1997) 217–223], I. Olkin proved the Craig–Sakamoto Theorem by basing his arguments on a determinantal lemma. This note presents a simple proof of the lemma that extends Olkin's result. © 1998 Elsevier Science Inc. All rights reserved.

Partition the $n$-square hermitian matrix $\mathbf{B}$ as

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} \tag{1}$$

in which $\mathbf{B}_{11}$ is $r$-square. For $1 \leqslant i < j \leqslant n$ let

$$\mathbf{B}(i,j) = b_{ii}b_{jj} - |b_{ij}|^2. \tag{2}$$

In [1] Olkin presented the following interesting Determinant Theorem. This note contains a simple proof of the Theorem and in fact establishes a somewhat stronger result.

**Theorem 1.** *If*

$$\operatorname{tr}(\mathbf{B}_{11}) = 0 \tag{3}$$

*and*

$$\sum_{1 \leqslant i<j \leqslant r} \mathbf{B}(i,j) + \sum_{1 \leqslant i \leqslant r, r+1 \leqslant j \leqslant n} \mathbf{B}(i,j) = 0 \tag{4}$$


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*then* $\mathbf{B}_{11}, \mathbf{B}_{12}$ *and* $\mathbf{B}_{21} = \mathbf{B}_{12}^*$ *are all zero matrices.*

**Proof.** Let $\lambda_1, \ldots, \lambda_r$ be the eigenvalues of $\mathbf{B}_{11}$. Then, from Eq. (3),

$$\begin{aligned} 0 &= (\lambda_1 + \cdots + \lambda_r)^2 \\ &= \lambda_1^2 + \cdots + \lambda_r^2 + 2E_2(\lambda_1, \ldots, \lambda_r) \\ &= \|\mathbf{B}_{11}\|^2 + 2\,\mathrm{tr}(C_2(\mathbf{B}_{11})). \end{aligned} \tag{5}$$

The norm is the Euclidean norm, $E_2(\lambda_1, \ldots, \lambda_r)$ is the second elementary symmetric function of $\lambda_1, \ldots, \lambda_r$, and $C_2(\mathbf{B}_{11})$ is the second compound matrix of $\mathbf{B}_{11}$. Note that the second sum on the left in Eq. (4) is equal to

$$\sum_{j=r+1}^{n} b_{jj}\left(\sum_{i=1}^{r} b_{ii}\right) - \|\mathbf{B}_{12}\|^2, \tag{6}$$

which from Eq. (3) is simply $-\|\mathbf{B}_{12}\|^2$. Thus, from Eq. (4),

$$\mathrm{tr}\,(C_2(\mathbf{B}_{11})) - \|\mathbf{B}_{12}\|^2 = 0 \tag{7}$$

and combining Eq. (5) with Eq. (7) we have

$$\begin{aligned} \|\mathbf{B}_{11}\|^2 &= -2\,\mathrm{tr}((C_2(\mathbf{B}_{11})) \\ &= -2\|\mathbf{B}_{12}\|^2. \end{aligned} \tag{8}$$

Hence $\mathbf{B}_{11}, \mathbf{B}_{12}$ and $\mathbf{B}_{21} = \mathbf{B}_{12}^*$ are zero matrices. □

Actually, the same conclusions about $\mathbf{B}$ follow if the equality sign in Eq. (4) is relaxed to " $\geqslant$ ". For, from Eqs. (5) and (6),

$$\begin{aligned} \mathrm{tr}(C_2(\mathbf{B}_{11})) - \|\mathbf{B}_{12}\|^2 &\geqslant 0, \\ \frac{-\|\mathbf{B}_{11}\|^2}{2} &\geqslant \|\mathbf{B}_{12}\|^2. \end{aligned}$$

## References

[1] I. Olkin, A determinantal proof of the Craig–Sakamoto Theorem, Linear Algebra Appl. 264 (1997) 217–223.